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1989 J. Phys. A: Math. Gen. 22 1223

(http://iopscience.iop.org/0305-4470/22/9/013)

J. Phys. A: Math. Gen. 22 (1989) 1223-1243. Printed in the UK

The Weyl orbits of G_2, F_4, E_6 and E_7

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Received 27 July 1988, in final form 15 December 1988

Abstract. Simple methods are described for constructing the Weyl orbits of exceptional Lie algebras and for identifying the orbit of an arbitrary weight. The methods make use of convenient bases in weight space. They are applied to G_2, F_4, E_6 and E_7. A complete table of all weights of all orbits of F_4 is given. The weights of the 37 orbits of smallest weight length of E_6 and E_7 are given in tabular form. The depth structures of the orbits are discussed.

1. Introduction

Any method of constructing irreducible representations (irreps) of simple Lie algebras must make use of Weyl reflection symmetry if it is to be efficient for large irreps. The construction problem may be separated into two parts. The first part consists of finding all the Weyl orbits in an irrep and their multiplicities. The second part consists of finding all the weights in each orbit. One of the earlier authors to emphasise this approach was Humphreys [1].

Simple general algorithms exist for the first part [2, 3]. Furthermore, extensive tables exist that list the multiplicities of all Weyl orbits in many irreps of all the simple Lie algebras [4]. The present paper is concerned with the second step, finding the weights in each orbit. A closely related problem, also discussed, is finding the orbit of any given weight.

For each of the classical algebras A_n, B_n, C_n and D_n, special orthogonal bases are known in which the rules for Weyl reflections are particularly simple [5, table 2]. In these bases the orbit corresponding to a particular weight may be obtained almost immediately. We are concerned here with the exceptional algebras, for which the situation is not quite so simple, It is helpful to use a basis appropriate to a classical subalgebra of the exceptional algebra in question. In a previous paper [3] the author used a basis that is natural for D_8 to construct many of the orbits of E_8. A fast procedure was given for finding the orbit of any given weight.

The main purpose of this paper is to extend the results of [3] to the exceptional algebras G_2, F_4, E_6 and E_7. The classical subalgebras used for the bases are, respectively, SU(3), SO(9), SU(3) × SU(3) × SU(3) and SU(8). The reasons for these choices, and a short discussion of other possible choices, are given in § 8. In the cases of G_2 and F_4, formulae for all weights of all orbits are given. The geometrical classification scheme proposed recently by the author [6] is used to help picture the structures of the orbits.

The numbers of subalgebra orbits in orbits of E_6 or E_7 are sufficiently large that one cannot list all weights of all orbits in short tables. The results for these two algebras are presented in two ways. First, a fast iterative procedure is given for finding the orbit of an arbitrary weight. Second, tables are given that list the weights of the 37 or more shortest orbits (orbits with shortest weights). The results for G_2, F_4, E_6 and E_7 are given in §§ 4-7.

The root sets of the exceptional algebras in these special bases are well known. The new features of the method are the fast procedures for identifying and constructing orbits. These procedures depend not only on the subalgebra, but also on the choice of an ordered set of orthogonal axes, and the consequent set of roots that are simple.

2. The method

Before discussing the method we review briefly some of the basic concepts and definitions that are used. Proofs may be found in the literature, for example, in Cahn [7]. We consider a simple Lie algebra of rank n. An ordered set of orthogonal axes is chosen in the n-dimensional weight space. A weight is defined as positive if its first non-zero component is positive. A simple root is a positive root that cannot be written as a sum of two positive roots. If R_i ($i=1$ to n) are the simple roots and M is an arbitrary weight, the Dynkin components m_i of M are defined by the scalar product equation

$$m_i = \langle R_i, M\rangle(2/R_i^2). \tag{2.1}$$

The length squared of any weight M is given in terms of its Dynkin components by the equation

$$M^2 = \sum_{ij} m_i m_j G_{ij}. \tag{2.2}$$

where G is the metric tensor, tabulated by Slansky [8, table 7].

The Weyl reflection S_α associated with the non-zero root α transforms the weight M into $S_\alpha(M)$, defined by

$$S_\alpha(M) = M - \langle\alpha, M\rangle(2/\alpha^2)\alpha. \tag{2.3}$$

If α is the simple root R_i, it is seen from (2.1) and (2.3) that

$$S_i(M) = M - m_i R_i. \tag{2.4}$$

All the weights that may be obtained from a weight M by sequences of zero or more Weyl reflections comprise the Weyl orbit of M.

A dominant weight is defined to be a weight with no negative Dynkin components. There is exactly one dominant weight in each Weyl orbit; this weight is used to characterise the orbit. Let M^{2+} be the dominant weight of the orbit T that contains the weight M. It is well known that the dimension D_T of the orbit is given by

$$D_T = D(G)/D(G_0) \tag{2.5}$$

where $D(X)$ is the order of the Weyl group of the algebra X and G_0 is the algebra that corresponds to the Dynkin diagram of G with all circles (and connecting lines) deleted that correspond to positive Dynkin components of M^{2+} [9]. If all m_i^{2+} are positive the orbit is called maximal, and $D(G_0)$ is defined to be one.

The bases used in this paper have been obtained by the replacement procedure described previously [10]. However, for simplicity, I do not refer to this procedure when defining the bases in §§ 4-7. Each basis is defined in the following way. For the nth rank exceptional algebra G under consideration, a classical subalgebra H of the same rank is specified. The roots of G consist of all the roots of H plus all the weights of some other representation of H.

The orientation of the ordered set of axes is then specified. This leads to the identification of the simple roots of G and H. The procedure is such that all but one of the simple roots of G are simple roots of H. The other simple root of G, which is not a root of H, is called the replacement root. The first axis is oriented to be orthogonal to the subspace generated by the $(n-1)$ simple roots of G that are roots of H. The first component of the replacement root is positive, of course.

In these bases every Weyl orbit of the exceptional algebra G is the union of one or more complete orbits of H. In order to understand how these bases simplify the construction of orbits, we examine first the standard construction method in the Dynkin basis. A simple reflection is defined as the Weyl reflection associated with a simple root. It is seen from (2.4) that the simple reflection S_i of a weight M leads to a more positive weight, more negative weight, or the same weight if the Dynkin component m_i is negative, positive or zero, respectively. A positive simple reflection series of a weight is defined as a series of simple reflections, each of which increases the positivity of the weight. If one wishes to find the Weyl orbit of a weight M the standard method is to apply a series of positive simple reflections until the dominant orbit weight is obtained. Similarly, the standard method of constructing the whole orbit from the dominant weight is by applying the possible negative simple reflection series. An example is given by Moody and Patera [9]. The disadvantage of these procedures is that the number of reflections in such a series may be as large as the number of positive roots in the algebra.

However, if one uses a basis of the type discussed here, all simple Weyl reflections are trivial except those associated with the replacement root. This shortens the construction procedure greatly. The construction is facilitated by the fact that the first orthogonal component of a weight measures the component of the replacement root, since this is the only simple root with a component in the first direction.

In some cases we will classify weights by using the geometrical classification parameters introduced in a previous reference [6]. For each weight M, the signature of a positive root π_i is defined to be negative if

$$\langle \pi_i, M\rangle < 0. \tag{2.6}$$

Otherwise the signature is positive. Each weight may be classified by the list of positive roots with negative signatures. The depth N of the weight is the number of π_i in the list. This depth is the same as the number of terms in a positive simple reflection series from M to M^{2+}.

Each orbit belongs to one of a finite number of patterns, where a pattern is defined by the set of Dynkin components of the dominant weight M^{2+} that are zero. The set of signature lists that denote the weights of an orbit is the same for all orbits of the same pattern. The depth of M^{2-}, the most negative weight of an orbit, is called the orbit depth and is given by

$$N(M^{2-}) = P(G) - P(G_0) \tag{2.7}$$

where $P(X)$ is the number of positive roots in the algebra X.

3. The SU(n) bases

Three of the bases used in this paper are based on SU(n) algebras; we discuss some general features of these bases here. The standard orthogonal basis for the $(n-1)$th rank algebra SU(n) involves the introduction of an extra, nth, dimension, and the projection of the weights on a particular $(n-1)$-dimensional subspace [11, appendix A]. In order to avoid this complication we use an alternative procedure. This procedure makes use of the fact that in order to determine simple roots and construct irreps and orbits, it is not necessary to know the exact orientation of the orthogonal axes.

The non-zero roots are of the type $(q\bar{r})$, where q and r are different weights of the fundamental representation N, and $\bar{r}=-r$ is the conjugate to r, a weight in the conjugate representation $\bar{N}$. The scalar products of these weights are

$$q^2=(n-1)/n \qquad \langle q, r\rangle = -1/n. \tag{3.1}$$

However positivity is defined, I number the weights of the irrep N in order of positivity, with 1 being the most positive. The positive roots are then those of the type $(j\bar{l})$ where $j<l$, and the simple roots are those with adjacent indices, i.e.

$$R_j=(j\overline{j+1}). \tag{3.2}$$

If λ_j are the Dynkin components of a weight Λ, n integers f_j are introduced that satisfy the equations

$$\lambda_j=f_j-f_{j+1}. \tag{3.3}$$

Clearly the λ_j are unchanged if all f_j are increased by the same amount. Hence, one additional condition must be used in conjunction with (3.3) to determine the f_j. I make the requirement that this condition be invariant to permutations of the f_j. One condition, used frequently, is $f_j(\text{minimum})=0$. If the weight Λ is dominant, then $f_j \geqslant f_{j+1}$, and f_j is the number of boxes in the j row of the Young tableau that represents the irrep with highest weight Λ. If one constructs a weight M from weights of the two fundamental representations, then f_j is the number of weights j minus the number of weights $\bar{j}$ in M. The advantage of the integers f_j (called here tableau components) is that they behave in a simple manner under Weyl reflections. It is straightforward to show that the reflection generated by the root $(j\bar{l})$ interchanges f_j and f_l and leaves all other f_i unaffected. Consequently, the Weyl orbit of a weight consists of all possible distinct permutations of the f_j; the dominant weight is the permutation that satisfies the condition $f_j \geqslant f_{j+1}$.

When SU(n) bases are used for exceptional algebras in §§ 4, 6 and 7, a further assumption concerning positivity is necessary. It turns out that it is sufficient to specify the orientation of the first orthogonal axis.

4. The algebra G_2

The subalgebra chosen is SU(3). The 14 roots of G_2 are the SU(3) roots plus the weights of the two fundamental representations 3 and $\bar{3}$. The properties of this G_2 basis are well known and the multiplicities of the weights in the G_2 irreps have been given by King and Qubanchi [12]. Therefore, I will give only a brief discussion of G_2, as an illustration of the technique discussed in § 2.

As in § 3, I label the weights of the irrep 3 in order of decreasing positivity A, B and C. The first orthogonal axis is chosen in the direction of the conjugate weight $\bar{C}$. The positive roots are then $(A\bar{B})$, $(B\bar{C})$, $(A\bar{C})$, A, B and $\bar{C}$. The first three (long) roots are taken of length $\sqrt{2}$, in which case the latter three (short) roots are of length $(2/3)^{1/2}$. The simple roots are

$$R_1=(A\bar{B}) \qquad \text{and} \qquad R_2=B. \tag{4.1}$$

The replacement root is the short root B.

The two SU(3) Dynkin components of a weight are denoted by λ_A and λ_B. These are related to the tableau components f_A, f_B and f_C (defined in § 3) by

$$\lambda_A=f_A-f_B \qquad \lambda_B=f_B-f_C. \tag{4.2}$$

It follows from (3.1) that the scalar products of the weights of the fundamental representation with an arbitrary weight M are

$$\langle A, M\rangle=\tfrac{1}{3}(2f_A-f_B-f_C)=\tfrac{1}{3}(2\lambda_A+\lambda_B) \tag{4.3a}$$

$$\langle B, M\rangle=\tfrac{1}{3}(2f_B-f_A-f_C)=\tfrac{1}{3}(\lambda_B-\lambda_A) \tag{4.3b}$$

$$\langle C, M\rangle=\tfrac{1}{3}(2f_C-f_A-f_B)=-\tfrac{1}{3}(2\lambda_B+\lambda_A). \tag{4.3c}$$

It follows from (2.1), (4.1) and (4.3a, b) that the G_2 Dynkin components m_1 and m_2 are related to the λ by

$$m_1=\lambda_A \qquad m_2=\lambda_B-\lambda_A. \tag{4.4}$$

The set of weights for an algebra is the set of vectors with integral Dynkin components. It is seen from (4.4) that the set of G_2 weights and the set of SU(3) weights are identical.

A Weyl orbit may be generated from any contained weight by simple reflections only. The only simple G_2 reflection that can connect different SU(3) orbits is that associated with R_2, the weight B. The SU(3) Dynkin components of B are $(-1\ 1)$. It follows from (2.3) and (4.3b) that the effect of the G_2 reflection S_2 on the SU(3) components of a weight is

$$S_2(\lambda_A\lambda_B)=(\lambda_B\lambda_A). \tag{4.5}$$

However, $(\lambda_B\lambda_A)$ is a weight in the SU(3) orbit conjugate to the orbit of $(\lambda_A\lambda_B)$. Therefore every self-conjugate SU(3) orbit is an entire G_2 orbit. If an SU(3) orbit O is not self-conjugate, the G_2 orbit is $O+O^*$. (In terms of weights of the fundamental SU(3) representations, the S_2 reflection is equal to the simultaneous transformations $B \leftrightarrows \bar{B}$ and $A \leftrightarrows \bar{C}$.)

Since G_2 is of second rank the Weyl reflection lines and enclosed sectors may be plotted in a plane. Each of the 12 sectors is an open wedge of width 30°. If one uses the basis discussed here, and chooses the positive first axis to be the upward vertical axis, the orientation of the diagram will correspond to the conventional orientation for the fundamental irrep of SU(3); the long root $(A\bar{B})$ will lie on the positive horizontal axis. The G_2 roots are plotted in many references, for example Georgi [13].

5. The algebra F_4

In the case of F_4 it is equally efficient to use as the subalgebra B_4[SO(9)] or C_4. I will use a B_4 basis because rotation groups are familiar; furthermore, if F_4 is a meaningful symmetry group of particle physics, it is likely that the rotation group distinction between tensors and spinors is physically significant. Recently, Neuberger has discussed F_4 [14]. The roots and co-roots of this reference are those obtainable from a C_4 basis, and a B_4 basis, respectively.

There is a well known, convenient orthogonal basis for B_4, in which the 24 long roots are $[\pm 1\ \pm 1\ 0\ 0]$ and the corresponding weights with components permuted, and the eight short roots are $[\pm 1\ 0\ 0\ 0]$ and the weights with components permuted [5]. I will use the shorthand notation $1_+3_- = [1\ 0\ {-1}\ 0]$, $2_- = [0\ {-1}\ 0\ 0]$, etc. The simple B_4 roots are

$$1_+2_-,\ 2_+3_-,\ 3_+4_- \text{ and } 4_+. \tag{5.1}$$

The weights are all vectors with integral Dynkin components. It follows from (2.1) and (5.1) that the B_4 weights are of two types. The tensor weights are all vectors such that each orthogonal component is integral. The spinor weights are all vectors such that each component is half-odd integral.

The effects of Weyl reflections are simple in the special basis. The symbol f_j is used to denote the jth orthogonal component. The Weyl transformation associated with the long root i_+k_- (or i_-k_+) leads to the component transposition $f_i \leftrightarrows f_k$, the reflection associated with the long root i_+k_+ (or i_-k_-) leads to the transformation $f_i \leftrightarrows -f_k$, and the reflection associated with the short root i_+ (or i_-) leads to $f_i \to -f_i$. Therefore, the Weyl orbit of a weight consists of all possible distinct permutations of the orthogonal components, with all possible sign combinations. It is seen from (2.1) and the list of simple roots (5.1) that for the dominant weight of a B_4 orbit all the f_i are non-negative and $f_i \geqslant f_{i+1}$.

The 24 long roots of F_4 are taken to be the 24 long roots of B_4, while the 24 short roots of F_4 are taken to be the 8 short roots of B_4 plus the 16 fundamental spinor weights $[\pm\frac{1}{2}\ \pm\frac{1}{2}\ \pm\frac{1}{2}\ \pm\frac{1}{2}]$. The simple roots of F_4 in this basis are those listed in figure 1. It follows from these roots and (2.1) that the Dynkin components m_i are related to the orthogonal components f_i by the equations,

$$\begin{aligned} m_1 &= f_2 - f_3 & m_2 &= f_3 - f_4 \\ m_3 &= 2f_4 & m_4 &= f_1 - f_2 - f_3 - f_4. \end{aligned} \tag{5.2}$$

The inverse equations are

$$\begin{aligned} f_1 &= m_1 + 2m_2 + \tfrac{3}{2}m_3 + m_4 & f_2 &= m_1 + m_2 + \tfrac{1}{2}m_3 \\ f_3 &= m_2 + \tfrac{1}{2}m_3 & f_4 &= \tfrac{1}{2}m_3. \end{aligned} \tag{5.3}$$

The first three F_4 simple roots are B_4 simple roots; the spinor root R_4 is the replacement root.

The procedure for finding the dominant weight of the F_4 orbit of an arbitrary weight M may now be described. If the Dynkin components of M are given, one uses (5.3) to find the orthogonal components. One then finds the dominant weight of the B_4

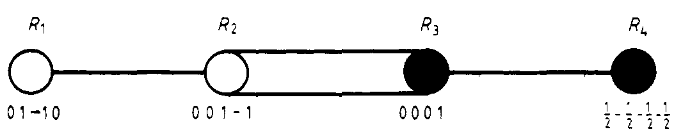

Figure 1. The simple roots of F_4 in the B_4 basis.

orbit of M by changing the signs of the negative orthogonal components and permuting the resulting components so that they satisfy $f_i \geq f_{i+1}$. The first three Dynkin components of this B_4-dominant weight must be non-negative. Equation (5.2) is used to find m_4; if $m_4 \geq 0$, the weight is the dominant weight of the F_4 orbit.

If $m_4 < 0$, one makes an S_4 reflection. The easiest way to do this is first to change the signs of the last three orthogonal components, calling the new components g, i.e. $g_1 = f_1$, $g_j = -f_j$ $(j > 1)$. One then transforms the g_i to g'_i, from the formula

$$g'_i = g_i - \tfrac{1}{2} m_4. \tag{5.4}$$

The reflection could be completed by changing the signs of the last three g'_i. However, this latter sign change is unnecessary, because we are interested only in the B_4 orbit of the g'. The dominant weight of the B_4 orbit of the g' must be the dominant weight of the F_4 orbit; i.e. one S_4 reflection is sufficient. (This may be verified by considering the elements of table 1, to be discussed shortly.)

Table 1. Weights of the Weyl orbits of F_4, in the B_4 basis. The F_4 Dynkin components m_1^{2+}, m_2^{2+}, m_3^{2+} and m_4^{2+} are denoted by a, b, c and d, respectively.

Class I $\{c>0, d>0\}$ (24, 23, 23, 21)				
(i)	$a+2b+\frac{3}{2}c+d(0)$	$a+b+\frac{1}{2}c(5)$	$b+\frac{1}{2}c(6a)$	$\frac{1}{2}c(7b)$
(ii)	$a+2b+\frac{3}{2}c+\frac{1}{2}d(1)$	$a+b+\frac{1}{2}c+\frac{1}{2}d(4)$	$b+\frac{1}{2}c+\frac{1}{2}d(5a)$	$\frac{1}{2}c+\frac{1}{2}d(6b)$
(iii)	$a+2b+c+\frac{1}{2}d(2)$	$a+b+c+\frac{1}{2}d(3b)$	$b+c+\frac{1}{2}d(4a)$	$\frac{1}{2}d(7)$
Class II $\{c>0, d=0\}$ (23, 22, 22, 20)				
(i)	$a+2b+\frac{3}{2}c(0)$	$a+b+\frac{1}{2}c(4)$	$b+\frac{1}{2}c(5a)$	$\frac{1}{2}c(6b)$
(ii)	$a+2b+c(2)$	$a+b+c(3b)$	$b+c(4a)$	$0(7)$
Class III $\{c=0, d>0\}$ (23, 20, 22, 15)				
(i)	$a+2b+d(0)$	$a+b(5)$	$b(6a)$	$0(7b)$
(ii)	$a+2b+\frac{1}{2}d(1)$	$a+b+\frac{1}{2}d(3b)$	$b+\frac{1}{2}d(4a)$	$\frac{1}{2}d(6b)$
Class IV $\{c=d=0\}$ (21, 15, 20, 0)				
(i)	$a+2b(0)$	$a+b(3b)$	$b(4a)$	$0(6b)$

Consider as an example the F_4 weight M with Dynkin components $(-8\ 4\ 2\ -1)$. From (5.3) the orthogonal components of this weight are $[2\ -3\ 5\ 1]$. (Square brackets are used for orthogonal components.) The dominant weight of the B_4 orbit of M is $[5\ 3\ 2\ 1]$. From (5.2) the value of m_4 is -1, so this weight is not F_4 dominant. An S_4 reflection is needed. We change the signs of the last three components, yielding $[5\ -3\ -2\ -1]$. We add $\frac{1}{2}|m_4|$ (subtract $\frac{1}{2}m_4$) to each of these components, yielding $[\frac{11}{2}\ -\frac{5}{2}\ -\frac{3}{2}\ -\frac{1}{2}]$. The dominant weight of the B_4 orbit of this weight is $[\frac{11}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2}]$. This weight is F_4 dominant. The Dynkin components are $(1\ 1\ 1\ 1)$.

If we use the classification parameters discussed in § 2, the positive roots π_i that satisfy $\langle \pi_i, M\rangle < 0$ are $[1\ 1\ 0\ 0]$, $[0\ 1\ 0\ 1]$, $[1\ 0\ -1\ 0]$, $[0\ 1\ -1\ 0]$, $[0\ 1\ 0\ -1]$, $[0\ 1\ 0\ 0]$, $[+\ +\ -\ +]$, $[+\ +\ -\ -]$ and $[+\ -\ -\ -]$, where $+$ and $-$ denote $\frac{1}{2}$ and $-\frac{1}{2}$. The depth of M is 9.

We next consider the problem of constructing the F_4 orbits from the dominant weights. The construction is easy for two reasons. First, since F_4 is a fourth-rank algebra, there are only $2^4 = 16$ patterns. Second, the maximum number of B_4 orbits in an F_4 orbit is three. It turns out that the expressions for the weights of an orbit may be placed in a form that depends only on whether m_3^{2+} and m_4^{2+} are zero or positive. Thus, there are four classes.

The dominant weights of all B_4 orbits in every F_4 orbit are given in table 1. The four components in each row correspond to a B_4 orbit, and are arranged so that $f_i \geqslant f_{i+1}$ and $f_i \geqslant 0$. It is seen that there are three B_4 orbits in an F_4 orbit only if both m_3^{2+} and m_4^{2+} are positive.

All numbers in ordinary parentheses refer to depths. The four successive numbers in parentheses following the class label are the depths of the orbits (computed from (2.7)), corresponding to the (a, b) patterns $(+\,+)$, $(+\,0)$, $(0\,+)$ and $(0\,0)$, respectively.

Before explaining the depths following the components, we note that it is convenient to separate the positive roots into two classes. Class one contains the roots for which $f_1 > 0$ and class two contains the roots for which $f_1 = 0$. The roots of class two are those of the B_3 of the components f_2, f_3 and f_4. The roots R_1, R_2 and R_3 are the simple roots of this B_3. The B_3 depths are easy to compute, so the depths in the table all refer to weights that are dominant with respect to this B_3.

The depth number following a component is the depth of the weight in which that component is along the first axis, and the other three are in order of decreasing magnitude. Thus, in orbit (iii) of class I, the (7) after the component $\frac{1}{2}d$ means the depth is seven of the weight $\frac{1}{2}d$, $a+2b+c+\frac{1}{2}d$, $a+b+c+\frac{1}{2}d$, $b+c+\frac{1}{2}d$. The $(4a)$ after the component $b+c+\frac{1}{2}d$ means the depth is four if that component is first and $a>0$. If $a=0$, that component is the same as the component to the immediate left, and the 4 should be neglected.

For example, consider the orbit $a=b=0$, $c=d=2$. This is of class I, and the dominant weights of the three B_4 orbits are 5111, 4222 and 3331. The numbers in parentheses indicate that the depths of the weights 5111, 1511, 4222, 2422, 3331 and 1333 are respectively, 0, 5, 1, 4, 2 and 7.

It is not difficult to calculate the depth structures of all orbits of F_4. These are listed in table 2. The symbols involving + and 0 are the $(m_1^{2+}, m_2^{2+}, m_3^{2+}, m_4^{2+})$ patterns. The underlined numbers are the orbit depths, and the numbers in parentheses are the orbit dimensions (numbers of contained weights). The numbers in the columns other than the first are the dimensions of the different depths. These dimensions satisfy the symmetry property

$$D[N] = D[N_{\max} - N] \tag{5.5}$$

where $D[N]$ is the dimension of depth N and $N_{\max}$ is the orbit depth $[N(M^{2-})]$. Therefore, it is sufficient to give the dimensions up to depth $\frac{1}{2}N_{\max}$ (when $N_{\max}$ is even) or to depth $\frac{1}{2}(N_{\max}-1)$ (when $N_{\max}$ is odd). It is seen that the depth structures have the 'spindle-shape' properties that are known for level structures of representations, i.e. (5.5) together with the condition $D(N+1) \geqslant (D(N))$, when $N < \frac{1}{2}N_{\max}$. (These properties are discussed on p 32 of [8].)

One can list all the weights of an irrep of F_4 by using table 1 together with a table of dominant weight multiplicities. For example, consider the irrep (0 0 0 2) of dimension 324. One finds from a multiplicity table [4]

$$(0\,0\,0\,2)_{\mathrm{I}} = \underline{1}(0\,0\,0\,2)_{24} + \underline{1}(0\,0\,1\,0)_{96} + \underline{3}(1\,0\,0\,0)_{24} + \underline{5}(0\,0\,0\,1)_{24} + \underline{12}(0\,0\,0\,0)_{1} \tag{5.6}$$

where the subscript I refers to the irrep, and the parentheses on the right refer to orbits. The underlined numbers are the multiplicities, and the numerical subscripts are the orbit dimensions, obtained from table 2. One can list all the weights in the irrep from

Table 2. Dimensionalities of depths of F_4 orbit patterns.

Patterns:	$+000$ $000+$	$0+00$ $00+0$	$+0+0$ $0++0$ $0+0+$	$++00$ $00++$	$+00+$	$+++0$ $++0+$ $+0++$ $0+++$	$++++$
Orbit depths:	$\underline{15}$	$\underline{20}$	$\underline{22}$	$\underline{21}$	$\underline{20}$	$\underline{23}$	$\underline{24}$
Orbit dimensions:	(24)	(96)	(288)	(192)	(144)	(576)	(1152)

Depth	Dimensions						
0	1	1	1	1	1	1	1
1	1	1	2	2	2	3	4
2	1	2	4	3	3	6	9
3	1	3	6	5	4	10	16
4	2	4	9	7	6	15	25
5	2	5	12	9	8	21	36
6	2	6	15	11	9	27	48
7	2	7	18	13	10	33	60
8		7	20	14	11	38	71
9		8	22	15	12	42	80
10		8	23	16	12	45	87
11			24			47	92
12							94

(5.6) and table 1. For example the orbit (0 0 1 0) is of class II in table 1 and contains the B_4 orbits $(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ and (1 1 1 0).

6. The algebra E_6

In the case of E_6 the basis chosen is based on the subalgebra $\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3) = \mathrm{SU}(3)^3$. One reason for this choice is that $\mathrm{SU}(3)^3$ is used to label states in most unified theories of fundamental particles that involve E_6. Arbitrary weights of the fundamental triplets of the three SU(3) are denoted by K, k and κ, respectively. The specific weights of these three triplets, each set given in order of decreasing positivity, are (ABC), (abc) and $(\alpha\beta\gamma)$, respectively.

The 78 roots of E_6 are taken to be the 24 roots of $\mathrm{SU}(3)^3$, plus the 54 states of the representation $(Kk\kappa)+(\overline{Kk\kappa})$. (In many physical models the representation $(K\bar{k}\bar{\kappa})+(\bar{K}k\kappa)$ is used [15]; the relation between this assignment and that used here is discussed at the end of this section.) In order to make the positivity of each E_6 root definite, we align the first orthogonal axis in the direction of the weight $\bar{C}$ of the $\bar{K}$ triplet. The positive roots that are not roots of $\mathrm{SU}(3)^3$ are then the $Kk\kappa$ weights that include either A or B, and the $\bar{K}\bar{k}\bar{\kappa}$ weights that include $\bar{C}$.

It is straightforward to show that the simple roots of E_6 are those listed on the Dynkin diagram of figure 2. The root $R_3 = Bc\gamma$ is the replacement root; the others are simple roots of $\mathrm{SU}(3)^3$. Since all $\mathrm{SU}(3)^3$ roots are E_6 roots, each E_6 orbit is the union of complete $\mathrm{SU}(3)^3$ orbits.

The scalar products of the $Kk\kappa$ weights and an arbitrary weight M may be determined from (4.3a, b, c) and the corresponding equations for the fundamental triplets (abc) and $(\alpha\beta\gamma)$. It follows from the simple roots of figure 2 and (2.1) and

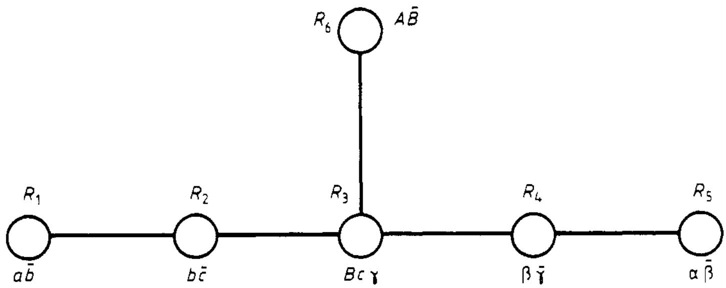

Figure 2. The simple roots of E_6 in the $\mathrm{SU}(3)^3$ basis.

(4.3a, b, c) that the E_6 Dynkin components m_i are related to the λ of the three SU(3) by the equations

$$\begin{aligned} &m_1=\lambda_a \qquad m_2=\lambda_b \\ &m_4=\lambda_\beta \qquad m_5=\lambda_\alpha \qquad m_6=\lambda_A \end{aligned} \tag{6.1}$$

$$m_3=\tfrac{1}{3}(\lambda_B-\lambda_A-\lambda_a-2\lambda_b-\lambda_\alpha-2\lambda_\beta). \tag{6.2}$$

The inverse equation for λ_B is given by

$$\lambda_B=m_1+2m_2+3m_3+2m_4+m_5+m_6. \tag{6.3}$$

The orbits and representations of each SU(3) may be grouped in three triality classes. For example, the triality of a K orbit is

$$\lambda_A+2\lambda_B(\mathrm{mod}\ 3) \qquad \text{or} \qquad \lambda_A-\lambda_B(\mathrm{mod}\ 3). \tag{6.4}$$

The weight set of E_6 is the set of vectors with integral Dynkin components. It is seen from (6.2) and (6.4) that m_3 is integral if and only if the sum of the trialities of the three SU(3) is zero (modulo 3). This implies that the weights of E_6 are the $\mathrm{SU}(3)^3$ orbits such that the trialities of the K, k and κ orbits are either all the same or all different.

There are also three triality classes for weights of E_6; the E_6 triality C is $(m_1+2m_2+m_4+2m_5)$, modulo 3 [16]. It is seen from (6.1) that $C=(\lambda_a+2\lambda_b)-(\lambda_\alpha+2\lambda_\beta)$, modulo 3. It follows from (6.4) that the E_6 triality is the SU(3) triality of the k multiplet, minus that of the κ multiplet. It follows that the weights of zero E_6 triality are those weights for which the three SU(3) trialities are the same. The weights of E_6 triality 1 are those weights for which the K, k and κ trialities are, respectively, (021), (210) or (102). The weights with E_6 triality 2 have K, k or κ trialities (012), (120) or (201).

Let X^* denote the SU(3) orbit conjugate to X, i.e. $(\lambda_1\lambda_2)^*=(\lambda_2\lambda_1)$, and let $(X\,Y Z)$ denote the $\mathrm{SU}(3)^3$ orbit in which X, Y and Z are the dominant weights of the orbits of the K, k and κ SU(3), respectively. Clearly, $(X^*\ Y^*\ Z^*)$ is in the E_6 orbit conjugate to that containing $(X\,YZ)$. Furthermore, it is well known that one may obtain E_6 conjugation by reflecting the Dynkin diagram, i.e. by making the simultaneous transpositions

$$m_1 \leftrightarrows m_5 \qquad m_2 \leftrightarrows m_4. \tag{6.5}$$

It is seen from (6.1) that the simultaneous transpositions of (6.5) are equivalent to transposing the k and κ SU(3). Therefore, $(X Z\ Y)$ is in the E_6 orbit conjugate to that of $(X\,YZ)$. However, one could have chosen the first orthogonal axis to be oriented parallel to $\bar{\gamma}$, rather than to $\bar{C}$. Then it would have been clear that (YXZ) and (XYZ) are in conjugate E_6 orbits.

These considerations lead to the conclusion that for any SU(3) orbits X, Y and Z, all distinct members of the following family of $SU(3)^3$ orbits are in the same E_6 orbit:

$$(XYZ), (YZX), (ZXY), (X^* Z^* Y^*), (Z^* Y^* X^*), (Y^* X^* Z^*). \tag{6.6}$$

Of course some of these orbits may be the same. The conjugate E_6 orbit contains the conjugate $SU(3)^3$ orbits, i.e. (XZY), etc. Clearly, if two of the X Y and Z are identical the E_6 orbit is self-conjugate.

We now consider the problem of finding the dominant weight of the E_6 orbit of an arbitrary weight M. A moderately efficient procedure is to follow literally the prescription given in [3]. One first finds the dominant weight of the $SU(3)^3$ orbit of M. Since all E_6 simple roots except R_3 are simple roots of $SU(3)^3$, the only Dynkin component that may be negative is m_3. If $m_3<0$, one makes an S_3 reflection and repeats the procedure until a non-negative m_3 is obtained.

This procedure can be made more efficient by using the following modification. After determining the dominant weight of the $SU(3)^3$ orbit of M, one examines the $SU(3)^3$ orbits related by (6.6), and considers the one whose dominant weight has the largest component along the first orthogonal axis. It is seen from (4.3c) that this component is proportional to $2\lambda_B+\lambda_A$. If there are two or more $SU(3)^3$ dominant weights in the family of (6.6) that have equally large values of $2\lambda_B+\lambda_A$, neither can be E_6 dominant; in such a case it is most efficient to choose among these weights one with a maximum value of $|m_3|$. When using this modified technique, I have not found a case in which more than two S_3 reflections are required to obtain E_6 dominance.

I will illustrate the method by finding the dominant weight of the E_6 orbit of the weight with E_6 Dynkin components $(6\ -5\ 7\ -4\ -3\ -2)$. By using (6.1) and (6.3) we find that the $SU(3)^3$ Dynkin components (λ) are $(-2\ 4)(6\ -5)(-3\ -4)$, where the order of the SU(3) corresponds to $Kk\kappa$. In order to find the dominant weight of the $SU(3)^3$ orbit, we use the tableau components f_i of (3.3), choosing one f_i for each SU(3) to have any convenient value. The result is [2 4 0] [6 0 5] [0 3 7]. The dominant $SU(3)^3$ weight is obtained by permuting the f of each SU(3), so that they satisfy $f_j \geq f_{j+1}$. The result is [4 2 0][6 5 0][7 3 0]. The Dynkin components of this weight are (2 2) (1 5) (4 3). The E_6 triality class is 1, and the length squared, obtained with the help of (2.2), is 160/3.

If we consider the $SU(3)^3$ dominant weights that are related to this weight by (6.6), there are two with the maximum value of 11 for $2\lambda_B+\lambda_A$. These are (1 5) (4 3) (2 2) and (3 4) (5 1) (2 2). We choose the latter weight, although the former is equally suitable. The m_3 value, determined from (6.2), is -4, so the weight is not E_6 dominant. An S_3 reflection must be made in which four times the root $Bc\gamma$ is added. This means that f_B, f_c and f_γ must be increased by 4. The result may be written as

$$[7\ 4+\underline{4}\ 0]\,[6\ 1\ \underline{4}]\,[4\ 2\ \underline{4}]. \tag{6.7}$$

Here the underlined numbers are the added numbers. The transformed weight of (6.7) is not $SU(3)^3$ dominant; the dominant weight of the $SU(3)^3$ multiplet is [8 7 0] [6 4 1] [4 4 2]. If one wishes, one may subtract any integer from the three f values of any SU(3). The $SU(3)^3$ Dynkin components of this weight are (1 7) (2 3) (0 2).

The value of m_3 for this weight is -2, so one must make another S_3 reflection. In the notation of (6.7) the new f values are $[8\ 7+\underline{2}\ 0]\ [6\ 4\ 1+\underline{2}]\ [4\ 4\ 2+\underline{2}]$. Again one permutes the f values in each SU(3) multiplet to obtain the dominant $SU(3)^3$ weight. The λ values are then

$$(1\,8)\,(2\,1)\,(0\,0). \tag{6.8}$$

This weight is the dominant weight of the E_6 orbit, with E_6 Dynkin components (2 1 1 0 0 1).

A related problem is that of constructing all the $SU(3)^3$ orbits in the E_6 orbit with a given dominant weight. One method of solving this problem is a generalisation of the method of [3]. One selects any convenient weight ψ of the $Kk\kappa + \bar{K}\bar{k}\bar{\kappa}$ representation, for example the weight $Aa\alpha$. Equations (6.1) and (6.3) are used to find the dominant $SU(3)^3$ orbit (orbit of the dominant E_6 weight). One performs an S_ψ reflection on all members of the dominant $SU(3)^3$ orbit, listing the resulting $SU(3)^3$ orbits together with the orbits related by (6.6). These new orbits are called primary orbits. One may use (2.5) to calculate the dimension of the E_6 orbit, and use this number to determine when all $SU(3)^3$ orbits have been found.

In some cases not all $SU(3)^3$ orbits may be obtained from one reflection of a weight in the dominant orbit. In these cases it is necessary to make S_ψ reflections of the weights in the primary orbits as well.

If one wishes to construct all E_6 orbits of a given length L, the following procedure is faster than that outlined above. First, all $SU(3)^3$ orbit families of length L are determined, where the family relation is (6.6). For each family one lists that dominant weight (or one of the dominant weights) with the largest first component, i.e. with the largest value of $2\lambda_B + \lambda_A$. One lists also the dominant weights of the E_6 orbits of length L, either by referring to published tables or by using (2.2). Equations (6.1) and (6.3) are used to identify the $SU(3)^3$-dominant weights that are E_6 dominant. One then applies an S_3 reflection to the other SU(3)-dominant weights, as illustrated above (6.7). If the $SU(3)^3$ orbits are considered in order of decreasing $2\lambda_B + \lambda_A$, then one S_3 reflection will serve to identify the E_6 orbit of each $SU(3)^3$ orbit. This follows because if a weight is $SU(3)^3$ dominant, and $m_3 < 0$, then an S_3 reflection will increase $2\lambda_B + \lambda_A$, and the resulting weight can be made $SU(3)^3$ dominant by reflections generated by simple roots other than R_3. These latter reflections do not change the value of $2\lambda_B + \lambda_A$. Finally, as a check, one adds the dimensions of the $SU(3)^3$ orbits in each E_6 orbit, and compares the sum with the result obtained from (2.5).

This method has been used here to calculate the $SU(3)^3$ content of all orbits of E_6 of length no greater than $(2\,0)^{1/2}$. Table 3 contains the E_6 orbits of triality zero. Each number with a wavy underline in the table represents three times the length squared of the orbit, while the symbol in curly brackets denotes the Dynkin components in shorthand form. Thus $\{1^25\}$ denotes the E_6 orbit with Dynkin components (2 0 0 0 1 0). The number following the curly brackets is the dimension of the orbit, expressed in prime factors, and the underlined number following that is the depth of the orbit.

Each of the $SU(3)^3$ orbit symbols $(\lambda_A\lambda_B)(\lambda_a\lambda_b)(\lambda_\alpha\lambda_\beta)$ represents the set of (one, two, three or six) distinct orbits in the family related by (6.6). The orbit given is one of the family that has the maximum value of $2\lambda_B + \lambda_A$. The underlined number following the orbit symbol is the depth of the weight given. The dominant $SU(3)^3$ orbit is listed first.

An orbit of E_6 triality zero that is not self-conjugate is denoted by an asterisk after its Dynkin symbol, i.e. {12}*. In such a case the conjugate orbit, obtained by making the diagram reflection of (6.5), is of the same length. Only one orbit of a conjugate pair is listed, since the other may be obtained by making the simultaneous transpositions

Table 3. E_6 Weyl orbits of triality 0 and length no greater than $(2\,0)^{1/2}$.

0 {0} 1 0	36 {1^3}* 3^3 16	48 {126}* $2^4 3^5 5$ 30	60 {156^2} $2^4 3^3 5$ 30
(00) (00) (00) 0	(03) (30) (00) 0	(14) (11) (00) 0	(24) (10) (10) 0
6 {6} $2^3 3^2$ 21	36 {24} $2^4 3^3 5$ 31	(04) (12) (01) 2	(33) (11) (00) 3
(11) (00) (00) 0	(04) (01) (01) 0	(23) (20) (01) 2	(04) (12) (12) 2
(01) (01) (01) 2	(13) (02) (10) 2	(13) (21) (02) 3	(13) (13) (02) 4
12 {15} $2 \cdot 3^3 5$ 24	(13) (10) (02) 2	(22) (11) (03) 4	(22) (03) (03) 6
(02) (10) (10) 0	(22) (11) (11) 3	54 {6^3} $2^3 3^2$ 21	60 {$1^3 6$}* $2^4 3^3$ 26
(11) (11) (00) 3	(03) (03) (00) 6	(33) (00) (00) 0	(14) (30) (00) 0
18 {3} $2^4 3^2 5$ 29	(12) (12) (20) 5	(03) (03) (03) 2	(04) (31) (01) 2
(03) (00) (00) 0	42 {$1^2 4$}* $2^3 3^3 5$ 29	54 {135} $2^4 3^4 5$ 33	60 {14^2}* $2^3 3^3 5$ 29
(12) (01) (01) 1	(04) (20) (01) 0	(05) (10) (10) 0	(05) (10) (02) 0
(02) (02) (10) 4	(13) (21) (10) 2	(14) (11) (11) 1	(14) (00) (03) 3
(11) (11) (11) 5	(22) (00) (03) 3	(23) (12) (01) 3	(23) (12) (20) 2
24 {12}* $2^4 3^3$ 26	(03) (30) (11) 5	(23) (01) (12) 3	(22) (03) (30) 6
(03) (11) (00) 0	42 {36} $2^5 3^2 5$ 30	(23) (20) (20) 4	
(12) (20) (01) 2	(14) (00) (00) 0	(04) (12) (20) 4	
24 {6^2} $2^3 3^2$ 21	(23) (01) (01) 1	(04) (20) (12) 4	
(22) (00) (00) 0	(13) (02) (02) 2	(13) (13) (10) 5	
(02) (02) (02) 2	(03) (03) (11) 4	(13) (21) (21) 5	
30 {156} $2^4 3^3 5$ 30	(12) (12) (12) 5	(22) (22) (11) 6	
(13) (10) (10) 0	48 {$1^2 5^2$} $2 \cdot 3^3 5$ 24	(03) (03) (30) 6	
(03) (11) (11) 2	(04) (20) (20) 0		
(22) (11) (00) 3	(22) (22) (00) 3		
(12) (12) (01) 4			
(12) (20) (20) 5			

Table 4. E_6 Weyl orbits of triality 1 and length less than $(2\,0)^{1/2}$.

4 {1} 3^3 16	34 {$5^2 6$} $2^4 3^3$ 26	46 {256} $2^5 3^3 5$ 32	52 {15^3} $2 \cdot 3^3 5$ 24
(01) (10) (00) 0	(13) (00) (20) 0	(14) (01) (10) 0	(04) (10) (30) 0
10 {4} $2^3 3^3$ 25	(03) (01) (21) 2	(04) (02) (11) 2	(13) (00) (31) 3
(02) (00) (01) 0	34 {13} $2^4 3^3 5$ 31	(23) (10) (11) 2	58 {124} $2^5 3^3 5$ 32
(11) (01) (10) 2	(04) (10) (00) 0	(23) (02) (00) 3	(05) (11) (01) 0
16 {5^2} 3^3 16	(13) (11) (01) 1	(13) (11) (12) 3	(14) (12) (10) 2
(02) (00) (20) 0	(03) (12) (10) 4	(13) (03) (01) 4	(14) (20) (02) 2
16 {16} $2^4 3^3$ 26	(03) (20) (02) 4	(22) (12) (02) 5	(23) (21) (11) 3
(12) (10) (00) 0	(22) (20) (10) 4	(03) (12) (21) 5	(23) (10) (03) 3
(02) (11) (01) 2	(12) (21) (11) 5	46 {$1^2 2$} $2^4 3^3$ 26	(04) (13) (00) 6
22 {25} $2^3 3^3 5$ 29	40 {16^2} $2^4 3^3$ 26	(04) (21) (00) 0	(13) (30) (12) 5
(03) (01) (10) 0	(23) (10) (00) 0	(13) (30) (01) 2	(13) (22) (20) 5
(12) (10) (11) 2	(03) (12) (02) 2	52 {$1^2 56$} $2^4 3^3 5$ 30	58 {35^2} $2^4 3^3 5$ 31
(12) (02) (00) 3	40 {2^2} $2^3 3^3$ 25	(14) (20) (10) 0	(05) (00) (20) 0
(02) (11) (20) 5	(04) (02) (00) 0	(04) (21) (11) 2	(14) (01) (21) 1
28 {$1^2 5$} $2 \cdot 3^3 5$ 24	(22) (20) (02) 2	(23) (21) (00) 3	(23) (10) (30) 4
(03) (20) (10) 0	40 {145} $2^4 3^3 5$ 30	(13) (22) (01) 4	(04) (02) (30) 4
(12) (21) (00) 3	(04) (10) (11) 0	(13) (30) (20) 5	(04) (10) (22) 4
28 {46} $2^3 3^3 5$ 29	(13) (11) (20) 2	52 {34} $2^4 3^3 5$ 31	(13) (11) (31) 5
(13) (00) (01) 0	(13) (00) (12) 3	(05) (00) (01) 0	58 {46^2} $2^3 3^3 5$ 29
(03) (01) (02) 2	(22) (12) (10) 3	(14) (01) (02) 1	(24) (00) (01) 0
(22) (01) (10) 2	(03) (20) (21) 5	(23) (02) (11) 2	(33) (01) (10) 2
(12) (02) (11) 3		(04) (10) (03) 4	(04) (02) (03) 3
		(13) (03) (20) 4	(13) (03) (12) 3
		(22) (12) (21) 5	

$\lambda_a \leftrightarrows \lambda_\alpha$, $\lambda_b \leftrightarrows \lambda_\beta$. For example, the orbit conjugate to {12} is {45}. The dominant weight of this orbit is (0 3) (0 0) (1 1).

Table 4 contains the corresponding information for the E_6 orbits of triality class 1. The conjugate orbit of any orbit of class 2 is of class 1, so it is unnecessary to provide a table for class 2.

For any weight M it is straightforward to find the positive roots π_i that satisfy $\langle \pi_i, M\rangle < 0$, and so contribute to the depth. For example, if the weight is (3 3) (1 1) (0 0) the positive roots are $Bc\dot{\alpha}$, $Bc\beta$ and $Bc\gamma$. The depths of some of the $\mathrm{SU}(3)^3$ dominant weights related by (6.6) to those listed in tables 3 and 4 are greater than those of the tables. For example, the depth of the weight (0 0) (3 3) (1 1) is 12.

An irrep may be constructed by making use of one of the tables of this section and a multiplicity table. I take as examples the irreps 351 {4} and 351′ $\{5^2\}$, where the curly brackets contain the Dynkin symbols in the notation explained earlier. These irreps are chosen because they occur in the direct product $27^* \times 27^*$ where 27^* is the irrep {5}. Therefore, as discussed by Rosner [17], Higgs particles in these irreps may contribute mass to fermions in the 27^* in a grand unified particle model.

A published multiplicity table yields the results [4],

$$\begin{aligned} \{4\}_\mathrm{I} &= \underline{1}\{4\}_{216} + \underline{5}\{1\}_{27} \\ \{5^2\}_\mathrm{I} &= \underline{1}\{5^2\}_{27} + \underline{1}\{4\}_{216} + \underline{4}\{1\}_{27} \end{aligned} \tag{6.9}$$

where the subscript I refers to the irrep, and the curly brackets on the right refer to orbits. The underlined numbers are the multiplicities, and the numerical subscripts are the orbit dimensions, obtained from table 4. One can list all the weights in the irreps from (6.9) and table 4.

In many models in which the algebra E_6 applies to fundamental particles, those roots of E_6 that are not roots of $\mathrm{SU}(3)^3$ are identified with the $\mathrm{SU}(3)^3$ representation $(3\,\bar{3}\,\bar{3}) + (\bar{3}\,3\,3)$ rather than $(3\,3\,3) + (\bar{3}\,\bar{3}\,\bar{3})$ [15]. The relation between the corresponding orbits of these two different $\mathrm{SU}(3)^3$ embeddings may be understood from the following construction. Let the E_6 roots that are not $\mathrm{SU}(3)^3$ roots be the weights $(K\bar{k}\bar{\kappa} + \bar{K}k\kappa)$. The first orthogonal axis is chosen to be in the direction of the A weight. The positive roots of E_6 that are not $\mathrm{SU}(3)^3$ roots are then the $K\bar{k}\bar{\kappa}$ weights that contain A and the $\bar{K}k\kappa$ weights that contain either of the $\bar{3}$ weights $\bar{B}$ or $\bar{C}$. It is straightforward to show that the simple roots, corresponding to the roots of the Dynkin diagram of figure 2, are

$$R_1 = (a\bar{b}) \qquad R_2 = (b\bar{c}) \qquad R_3 = (\bar{B}c\gamma)$$
$$R_4 = (\beta\bar{\gamma}) \qquad R_5 = (\alpha\bar{\beta}) \qquad R_6 = (B\bar{C}).$$

It is seen from these equations and (4.3*a*, *b*, *c*) that the relations between the E_6 and $\mathrm{SU}(3)^3$ Dynkin components may be obtained by making the transpositions $\lambda_A \leftrightarrows \lambda_B$ in (6.1) and (6.2). Therefore, the orbits may be obtained by transposing the two Dynkin components of the first SU(3) in tables 3 and 4 and (6.6).

7. The algebra E_7

In the case of E_7 the subalgebra is chosen to be $\mathrm{SU}(8) = A_7$. I use the notation of § 3, numbering the weights of the fundamental octet representation of SU(8) 1–8 in order of decreasing positivity. The roots of E_7 are taken to be the 63 roots of SU(8) plus the 70 weights of $\mathscr{A}_4$, the completely antisymmetric combination of four weights of the fundamental octet. The weights of $\mathscr{A}_4$ are denoted by listing the four octet weights that are present, i.e. $(ijkl)$. In order to make the positivity of all these weights definite, I make the further assumption that the first orthogonal axis is oriented in the direction of the octet weight 1. The positive weights of $\mathscr{A}_4$ are then those that contain weight 1. It is straightforward to show that the simple E_7 roots are those listed on the Dynkin diagram of figure 3. The $\mathscr{A}_4$ weight $R_7=(1\,6\,7\,8)$ is the replacement root. All other E_7 simple roots are SU(8) simple roots.

It follows from figure 3, (2.1) and (3.1) that the E_7 Dynkin components m_i are related to the SU(8) Dynkin components λ_i by

$$\begin{aligned} &m_1=\lambda_7 \qquad m_2=\lambda_6 \qquad m_3=\lambda_5 \\ &m_4=\lambda_4 \qquad m_5=\lambda_3 \qquad m_6=\lambda_2 \end{aligned} \tag{7.1}$$

$$m_7=\tfrac{1}{2}(\lambda_1-\lambda_3-2\lambda_4-3\lambda_5-2\lambda_6-\lambda_7). \tag{7.2}$$

The expression for m_7 is simpler when written in terms of the tableau components of (3.3), i.e.

$$m_7=\tfrac{1}{2}(f_1+f_6+f_7+f_8-f_2-f_3-f_4-f_5). \tag{7.3}$$

The inverse equation for λ_1 is

$$\lambda_1=m_1+2m_2+3m_3+2m_4+m_5+2m_7. \tag{7.4}$$

Since all SU(8) roots are E_7 roots, every weight of E_7 must be an SU(8) weight. It is required further that m_7 be an integer. It is seen from (7.3) that this implies $F=2k$,

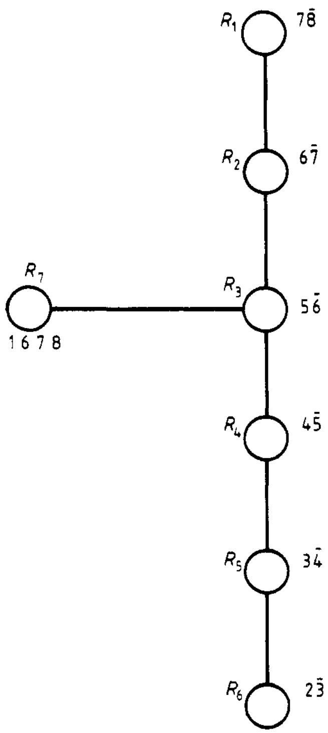

Figure 3. The simple roots of E_7 in the SU(8) basis.

where k is an integer and

$$F=\sum_{i=1}^{8} f_i. \tag{7.5}$$

The octality congruence number of SU(8) is F modulo 8, so the weights of E_7 are the SU(8) weights of even octality class.

Since R_7 is of octality class 4, weights differing by 4 in octality may be in the same E_7 orbit. Therefore, there are two E_7 congruence classes, duality-0 weights that have SU(8) octalities of either 0 or 4, and duality-1 weights that have octalities of either 2 or 6. It is well-known that the E_7 duality corresponds to $m_4+m_6+m_7$, modulo 2 [16]. In order to demonstrate that these two duality definitions are equivalent, we note that the SU(8) octality C may be written in terms of the λ in the following way:

$$C=F-8f_8=\sum_{j=1}^{7} j\lambda_j \text{ modulo } 8.$$

Since the root R_7 is of octality 4, we are interested in C modulo 4. If (7.1) and (7.4) are used, C modulo 4 may be written

$$2(m_4+m_6+m_7) \text{ modulo } 4.$$

Therefore, the even and odd values of $m_4+m_6+m_7$ do correspond to the two duality classes as defined in the SU(8) basis.

Ordinary parentheses and square brackets will be used, respectively, for SU(8) Dynkin components λ_i and tableau components f_i. As discussed in § 3 some convention must be used to determine one f_i. The convention used in this section is

$$F=0, 2, 4 \text{ or } -2 \tag{7.6}$$

for weights of octality 0, 2, 4 or 6, respectively.

If λ_i are the Dynkin components of the dominant weight of an SU(8) orbit, the components λ_i^* of the dominant weight of the conjugate orbit are given by $\lambda_i^*=\lambda_{8-i}$. For dominant weights of the octality classes 0, 2 and -2, the corresponding relation for the tableau components is

$$f_i^*=-f_{9-i}. \tag{7.7a}$$

For orbits of octality 4, the convention that $F=4$ implies that the relation is

$$f_i^*=-f_{9-i}+1. \tag{7.7b}$$

Every orbit of E_7 is self-conjugate, so conjugate SU(8) orbits necessarily are in the same E_7 orbit.

We next consider the problem of finding the dominant weight of the E_7 orbit of an arbitrary weight M. The method used is similar to that of § 6. One writes the weight in the SU(8) basis and determines the dominant weight of the SU(8) orbit. All E_7 simple roots except R_7 are SU(8) simple roots, so only the Dynkin component m_7 may be negative. If $m_7<0$, one makes an S_7 Weyl reflection, and continues. The procedure may be followed for any SU(7) orbit. However, one can improve the efficiency by considering each SU(8) orbit together with its conjugate orbit. If the dominant weight of either of these two orbits has a larger first orthogonal component than that of the other, one applies the process to the weight with larger first component. The first component is proportional to the quantity

$$f_1-\tfrac{1}{8}F. \tag{7.8}$$

I will illustrate the procedure by finding the dominant orbit weight of the weight M with E_7 Dynkin components $(8\ 1\ -9\ 8\ 0\ -6\ 2)$. By using (7.1) and (7.4) we find the SU(8) Dynkin components λ_i are $(3\ -6\ 0\ 8\ -9\ 1\ 8)$. In order to determine the tableau components, one chooses f_1 or f_8 arbitrarily, uses (3.3), and then adds or subtracts the appropriate integer from each f_i so that F is 0, 4, 2 or -2, as required by (7.6). The result is $[0\ -3\ 3\ 3\ -5\ 4\ 3\ -5]$. Since $F=0$ the weight is of E_7 duality class 0. One may use (3.1) to find that the length squared is 102.

The dominant weight of the SU(8) orbit is obtained by ordering the f_i by decreasing value. This weight is $[4\ 3\ 3\ 3\ 0\ -3\ -5\ -5]$. It is seen from (7.7*a*) and (7.8) that the first component is larger for the dominant weight of the conjugate orbit, so we consider this weight, $[5\ \underline{5\ 3\ 0\ -3}\ -3\ -3\ -4]$, underlining components f_2 through f_5 for convenience. The component m_7, calculated by using (7.3), is -5. Since this is negative, the weight is not E_7 dominant. An S_7 reflection must be made. The reflection may be made by adding $\frac{1}{2}|m_7|$ to the components (1 6 7 8) and subtracting $\frac{1}{2}|m_7|$ from the underlined components (2 3 4 5). If $|m_7|$ were even this would be the best procedure. However, for odd $|m_7|$ this procedure introduces fractions, and leads inevitably to adding (or subtracting) some number from all f_i to comply with the convention of (7.6). For $|m_7|$ odd it is easier to write $|m_7|=a+b$, where a and b are adjacent integers, and add a to the components (1 6 7 8) and subtract b from the components (2 3 4 5). If we add 3 to the (1 6 7 8) components and subtract 2 from the others, we obtain the weight $[8\ \underline{3\ 1\ -2\ -5}\ 0\ 0\ -1]$, a weight of octality 4. The dominant weight of the SU(8) orbit is $[8\ \underline{3\ 1\ 0\ 0}\ -1\ -2\ -5]$. For this weight $m_7=-2$, so E_7 dominance has not been obtained. We make a second S_7 reflection by adding 1 to the components not underlined and subtracting 1 from the underlined components. The resulting weight is $[9\ \underline{2\ 0\ -1\ -1}\ 0\ -1\ -4]$. The dominant weight of this SU(8) orbit is

$$[9\ 2\ 0\ 0\ -1\ -1\ -1\ -4]. \tag{7.9}$$

This weight is E_7 dominant, since $m_7=1$. The SU(8) Dynkin components are (7 2 0 1 0 0 3), and the E_7 Dynkin components, obtained by using (7.1) and (7.2), are (3 0 0 1 0 2 1). I have not found a weight such that this procedure requires more than two S_7 reflections.

If one wishes to find all the SU(8) orbits in an E_7 orbit, or to find all weights in all orbits of a given length, one may use procedures analogous to those described in § 6.

The SU(8) orbits contained in each E_7 orbit of duality class 0 and of length no greater than $(2\ 0)^{1/2}$ are listed in table 5. The numbers with wavy underlines are twice the length squared of the E_7 orbits; the symbols following these have the same meaning as in tables 3 and 4. The dominant weights of the SU(8) orbits are given in terms of the f components. If the SU(8) orbit is not self-conjugate, only one of the conjugate pair is given, always one with a maximum value of the first orthogonal component. The underlined number following each SU(8)-dominant weight is the depth of the weight. If the SU(8) orbit is not self-conjugate, a second underlined number gives the depth of the dominant weight of the conjugate orbit.

Table 6 contains the corresponding information for the E_7 weights of duality class 1. In this case none of the SU(8) orbits is self-conjugate.

For any weight M it is easy to determine the positive roots π_i that satisfy the condition $\langle \pi_i, M\rangle < 0$, and so contribute to the depth. The scalar product of the $\mathscr{A}_4$ root (1 6 7 8) with a weight is given by (7.3); corresponding equations apply to other weights of $\mathscr{A}_4$. It follows that the positive root $(1\ ijk)$ satisfies the condition $\langle 1\ ijk, M\rangle <$ 0 if and only if $f_1+f_i+f_j+f_k<\frac{1}{2}F$. For example, if we take M to be the weight

Table 5. E_7 Weyl orbits of duality 0 of length no greater than $(2\,0)^{1/2}$.

0 {0} 1 0	24 {3} $2^5 3^2 5 \cdot 7$ 53	36 {1^3} $2 \cdot 3^2 7$ 33
0 0 0 0 0 0 0 0 0	3 0 0 0 0 −1 −1 −1 0, 14	3 0 0 0 0 0 0 −3 0
4 {1} $2 \cdot 3^2 7$ 33	3 1 1 1 0 0 −1 −1 2, 9	2 2 2 2 −1 −1 −1 −1 4
1 0 0 0 0 0 0 −1 0	2 1 1 0 0 −1 −1 −2 5	36 {56^2} $2^3 3^3 7$ 43
1 1 1 1 0 0 0 0 4	2 2 1 1 1 −1 −1 −1 10, 9	3 2 0 −1 −1 −1 −1 −1 0, 9
8 {5} $2^2 3^3 7$ 42	28 {7^2} $2^6 3^2$ 42	3 3 0 0 0 0 −1 −1 4, 5
2 1 1 0 0 0 0 0 0, 9	4 0 0 0 0 0 0 0 0, 15	36 {25} $2^5 3^3 5 \cdot 7$ 55
1 1 0 0 0 0 −1 −1 4	3 1 1 1 1 −1 −1 −1 1, 6	4 1 1 0 0 0 −1 −1 0, 12
12 {2} $2^5 3^2 7$ 47	28 {46} $2^6 3^3 7$ 52	3 1 1 0 −1 −1 −1 −2 3, 9
2 0 0 0 0 0 −1 −1 0, 10	3 1 0 0 −1 −1 −1 −1 0, 12	3 1 0 0 0 0 −2 −2 4, 10
2 1 1 1 0 0 0 −1 3	3 2 1 0 0 0 −1 −1 3, 8	3 2 2 0 0 −1 −1 −1 7, 11
1 1 1 0 0 −1 −1 −1 7	2 2 0 0 0 −1 −1 −2 7, 5	3 2 1 1 0 0 −1 −2 5
12 {6^2} $2^3 7$ 27	2 2 1 −1 −1 −1 −1 −1 10, 13	2 2 1 0 0 −1 −2 −2 8
2 2 0 0 0 0 0 0 0, 5	28 {12} $2^6 3^2 7$ 48	40 {47} $2^6 3^2 5 \cdot 7$ 54
16 {67} $2^6 3^2 7$ 48	3 0 0 0 0 0 −1 −2 0, 10	4 0 0 0 −1 −1 −1 −1 0, 15
3 1 0 0 0 0 0 0 0, 15	3 1 1 1 0 0 0 −2 3	4 1 1 1 0 −1 −1 −1 1, 12
2 1 0 0 0 −1 −1 −1 1, 8	2 1 1 1 −1 −1 −1 −2 4	3 1 1 0 0 −1 −2 −2 3, 8
2 1 1 1 1 0 −1 −1 5, 6	2 2 2 1 0 −1 −1 −1 7	3 2 1 1 1 −1 −1 −2 7, 6
16 {1^2} $2 \cdot 3^2 7$ 33	32 {167} $2^7 3^3 7$ 53	2 1 1 1 1 −2 −2 −2 13, 10
2 0 0 0 0 0 0 −2 0	4 1 0 0 0 0 0 −1 0, 15	40 {26^2} $2^6 3^3 7$ 52
1 1 1 1 −1 −1 −1 −1 4	3 1 0 0 0 −1 −1 −2 1, 8	4 2 0 0 0 0 −1 −1 0, 11
20 {15} $2^3 3^3 5 \cdot 7$ 50	3 1 1 1 1 0 −1 −2 5, 6	3 2 0 0 −1 −1 −1 −2 3, 6
3 1 1 0 0 0 0 −1 0, 9	3 2 1 1 0 −1 −1 −1 4, 9	3 1 1 1 1 1 −2 −2 5, 10
2 1 0 0 0 0 −1 −2 4	2 1 1 1 0 −1 −2 −2 7, 7	3 3 1 0 0 −1 −1 −1 7, 9
2 1 1 0 −1 −1 −1 −1 4, 10	32 {5^2} $2^2 3^3 7$ 42	40 {$1^2 5$} $2^3 3^3 5 \cdot 7$ 50
2 2 1 1 0 0 −1 −1 6	3 1 1 −1 −1 −1 −1 −1 0, 9	4 1 1 0 0 0 0 −2 0, 9
24 {16^2} $2^3 3^3 7$ 43	2 2 0 0 0 0 −2 −2 4	3 1 0 0 0 0 −1 −3 4
3 2 0 0 0 0 0 −1 0, 5		3 2 2 1 −1 −1 −1 −1 4, 10
2 2 0 0 −1 −1 −1 −1 4, 7		2 2 1 1 −1 −1 −2 −2 6

[2 2 1 1 −1 −1 −2 −2] of the length-$(2\,0)^{1/2}$ orbit {$1^2 5$}, the positive roots that satisfy the condition are (1 3 7 8), (1 4 7 8), (1 5 6 7), (1 5 6 8), (1 5 7 8) and (1 6 7 8).

One may construct an E_7 irrep by using table 5 or 6 and a multiplicity table. I take for an example the 912-dimensional irrep {7}, since fermions are associated with this irrep in one model of particles [18]. Using a multiplicity table [4] one writes

$$\{7\}_1 = \underline{1}\{7\}_{576} + \underline{6}\{6\}_{56} \tag{7.10}$$

where the notation is the same as in (6.9). Table 5 and (7.10) may be used to write the weights of the irrep.

8. The choices of the subalgebras *H*

In this section the reasons are given for the specific choices of the subalgebras *H* used in §§4, 5, 6 and 7. The method developed in [3] and used here requires that *H* satisfy three criteria, listed below.

(1) *H* must be a classical algebra, or the direct product of a classical algebra and G_2, so that the orbit of *H* of an arbitrary weight may be determined quickly.

(2) *H* must be a regular subalgebra. If it is not, then one cannot find a set of simple roots of *G*, all but one of which are simple roots of *H*. In such a case the

Table 6. E_7 Weyl orbits of duality 1 and length less than $(2\,0)^{1/2}$.

3 {6} $2^3 7$ 27	27 {57} $2^6 3^3 7$ 52	35 {27} $2^6 3^3 7$ 52
1 1 0 0 0 0 0 0 0, 5	3 0 0 −1 −1 −1 −1 −1 0, 15	4 0 0 0 0 0 −1 −1 0, 15
7 {7} $2^6 3^2$ 42	3 1 1 0 0 −1 −1 −1 1, 10	3 0 0 0 0 −1 −2 −2 1, 10
2 0 0 0 0 0 0 0 0, 15	2 1 0 0 0 −1 −2 −2 4, 7	3 1 1 1 0 −1 −1 −2 3, 6
1 0 0 0 0 −1 −1 −1 1, 6	2 1 1 −1 −1 −1 −1 −2 6, 9	2 1 1 0 0 −2 −2 −2 7, 8
11 {16} $2^3 3^3 7$ 43	27 {$1^2 6$} $2^3 3^3 7$ 43	35 {156} $2^4 3^3 5 \cdot 7$ 51
2 1 0 0 0 0 0 −1 0, 5	3 1 0 0 0 0 0 −2 0, 5	3 1 0 −1 −1 −1 −1 −2 0, 9
1 1 0 0 −1 −1 −1 −1 4, 7	2 2 1 1 −1 −1 −1 −1 4, 7	3 2 0 0 0 0 −1 −2 4, 5
15 {4} $2^6 3^2 7$ 50	27 {6^3} $2^3 7$ 27	3 2 1 0 −1 −1 −1 −1 4, 10
2 0 0 0 −1 −1 −1 −1 0, 12	2 2 −1 −1 −1 −1 −1 −1 0, 5	2 2 0 0 −1 −1 −2 −2 6, 7
2 1 1 0 0 0 −1 −1 3, 7	31 {$6^2 7$} $2^6 3^2 7$ 48	39 {$1^2 7$} $2^6 3^2 7$ 48
1 1 1 −1 −1 −1 −1 −1 10, 13	3 1 −1 −1 −1 −1 −1 −1 0, 15	4 0 0 0 0 0 0 −2 0, 15
19 {17} $2^6 3^2 7$ 48	3 2 0 0 0 −1 −1 −1 1, 8	3 0 0 0 0 −1 −1 −3 1, 6
3 0 0 0 0 0 0 −1 0, 15	2 2 0 −1 −1 −1 −1 −2 6, 5	2 1 1 1 −1 −2 −2 −2 4, 7
2 0 0 0 0 −1 −1 −2 1, 6	31 {14} $2^6 3^2 5 \cdot 7$ 54	39 {36} $2^7 3^2 5 \cdot 7$ 56
2 1 1 1 0 −1 −1 −1 4, 7	3 0 0 0 −1 −1 −1 −2 0, 12	4 1 0 0 0 −1 −1 −1 0, 14
19 {56} $2^3 3^3 7$ 43	3 1 1 0 0 0 −1 −2 3, 7	3 1 0 0 −1 −1 −2 −2 2, 10
2 1 0 −1 −1 −1 −1 −1 0, 9	3 1 1 1 −1 −1 −1 −1 4, 13	3 2 1 0 0 −1 −1 −2 5, 7
2 2 0 0 0 0 −1 −1 4, 5	2 1 1 0 −1 −1 −2 −2 5, 8	3 1 1 1 0 0 −2 −2 5, 9
23 {26} $2^6 3^3 7$ 52	2 2 2 0 −1 −1 −1 −1 10, 13	2 2 0 0 0 −2 −2 −2 10, 11
3 1 0 0 0 0 −1 −1 0, 11		2 2 1 −1 −1 −1 −2 −2 9, 11
2 0 0 0 0 0 −2 −2 5, 10		
2 1 0 0 −1 −1 −1 −2 3, 6		
2 2 1 0 0 −1 −1 −1 7, 9		

process of finding the dominant weight of an orbit of G of an arbitrary weight would be lengthened, in general.

(3) The rank of H should be equal to the rank of G, exclusive of U_1 factors. When this requirement is met, finding the dominant H-orbit weight often leads to a weight of higher positivity than would result simply from making Weyl reflections associated with the simple roots of G that are simple roots of H. This point is illustrated in [3]. For each exceptional group, these three requirements limit the number of candidates for H to a small number.

For each basis considered one can assign an efficiency number $\mathscr{E}$, defined to be the maximum number of entries needed for an orbit of the exceptional algebra G in a table such as those of §§ 6 and 7. Clearly, it is desirable that $\mathscr{E}$ be as small as possible. It can be shown that $\mathscr{E}$ is the number of entries present for a maximal orbit of G, and is given by the formula

$$\mathscr{E} = \frac{D(G)}{N_F D(H)}$$

where $D(X)$ is the order of the Weyl group for the algebra X and N_F is the maximum number of H orbits related by a simple symmetry relation of the type of (6.6). N_F is 6 for the E_6 basis used here, but in most cases it is 1 or 2. If all orbits of G are self-conjugate, and some orbits of H are not self-conjugate, then N_F is at least 2.

It is clear that the SU(3) basis used for G_2 in § 4 is optimal, since the three conditions discussed above are met and the efficiency is 2. I list below the efficiencies corresponding to several possible choices of H for each of the other four exceptional algebras. For illustrative purposes, some bases are included that do not satisfy all three of the requirements listed above. Subalgebras that satisfy the three requirements are denoted with an asterisk. The number in parentheses is the efficiency. The first subalgebra listed for each G is the one used in this paper (or in the treatment of E_8 in [3]):

$G = F_4$ case: $B_4^*(3)$, $C_4^*(3)$, $C_3 \times A_1^*(12)$, $G_2 \times A_1(48)$

$G = E_6$ case $A_2^{3*}(40)$, $A_5 \times A_1^*(36)$, $D_5 \times U_1(27)$, $F_4(45)$

$G = E_7$ case: $A_7^*(36)$, $D_6 \times A_1^*(63)$, $A_5 \times A_2^*(336)$, $E_6 \times U_1(28)$

$G = E_8$ case: $D_8^*(135)$, $A_8^*(960)$, $E_7 \times A_1(120)$.

Some subalgebras that satisfy the three requirements have been omitted from the lists; in these cases the efficiencies are many times larger than those of the subalgebras chosen.

In the cases of E_7 and E_8 it is seen that of the bases satisfying the three requirements, the chosen basis is significantly more efficient than the others. Bases involving exceptional subalgebras may be useful in particular models involving broken symmetry. For example, if E_8 is broken to E_7, the $E_7 \times A_1$ basis may be useful. However, since the Weyl orbits of E_7 are not transparent, the effective efficiency in such a case is 120 times the efficiency of the basis used for E_7.

In the case of F_4 the B_4 and C_4 bases are significantly more efficient than any other. The reasons for choosing the B_4 basis are discussed at the beginning of § 5. In the case of E_6 the first two bases listed satisfy the three requirements and are of comparable efficiency. The A_2^3 basis was chosen because this subalgebra is used in many theories of particles, as discussed in § 6. The $A_5 \times A_1$ basis is useful, and has been studied by King and Al-Qubanchi [19].

In a particle model that involves $E_n \rightarrow E_{n-1}$ symmetry breaking, it might be convenient to use more than one basis. I illustrate by considering the case $n = 7$. One might employ the A_7 basis of figure 3 and also an $E_6 \times U_1$ basis. This latter basis may be obtained by following the prescription of § II of [10], i.e. by identifying the E_6 simple roots with all the roots of figure 3 except R_6, and assigning these roots to the dimensions 2-7. If the R_5 of figure 3 is the fifth simple root of E_6, one chooses for R_6 the weight $\{x + (0\,0\,0\,0\,-1\,0)\}$, where the numbers in parentheses are E_6 Dynkin components and x is a vector parallel to the positive first axis. The E_6 weight is of the same length as $(1\,0\,0\,0\,0\,0)$; it is seen from table 4 that this length is $(\frac{4}{3})^{1/2}$. Therefore, the length of x is $(\frac{2}{3})^{1/2}$, so that the E_7 root R_6 is of length $\sqrt{2}$. If one then uses the basis of § 6 for E_6, it is seen from (6.1) and (6.3) that the weight $(0\,0\,0\,0\,-1\,0)$ is $(C\bar{\alpha})$. With this basis the two expressions for the roots R_1-R_7 of figure 3 are

$$7\bar{8} = a\bar{b},\ 6\bar{7} = b\bar{c},\ 5\bar{6} = Bc\gamma,\ 4\bar{5} = \beta\bar{\gamma},\ 3\bar{4} = \alpha\bar{\beta},\ 2\bar{3} = x + (C\bar{\alpha}) \qquad (8.1a)$$

and

$$1\,6\,7\,8 = A\bar{B}. \qquad (8.1b)$$

These equalities are not consistent with the exact assumptions made concerning the orientation of the orthogonal axes in various sections of this paper. However, the orientation assumptions are used only to find the simple roots, and then may be disregarded. Therefore, one can use the equalities of (8.1) to analyse an $E_7 \rightarrow E_6$ model.

A similar technique may be used in a model with $E_8 \rightarrow E_7$ symmetry breaking.

Acknowledgement

This paper was supported in part by the US Department of Energy.

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